PYTHAGOREAN TRIPLETS

by **Tofique Fatehi**
(January 2009)

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1. ABBREVIATIONS:

AR	Aspect Ratio (C/B)	(2.3)
HN	Hypotenuse Number	(2.6)
PHN	Prime Hypotenuse Number	(2.7)
PPT	Primitive Pythagorean Triplet	(2.4)
PT	Pythagorean Triplet	(2.1)

2. DEFINITIONS:

2.1 A Pythagorean Triplet (PT) is a set of three positive, non-zero integers, A, B and C such that A^2 = B^2 + C^2. (Equation A). Geometrically this represents a right-angled triangle with A as the hypotenuse, and B and C the sides adjacent to the right angle. A PT is designated henceforth as {ABC}

2.2 To avoid unnecessary duplication, it is considered that B > C.

2.3 The ratio of C/B is termed the Aspect Ratio (AR) of the PT. Its values lies between 0 and 1.

2.4 If {ABC} do not have any common factors, then the PT is said to be a Primitive Pythagorean Triplet (PPT).

2.5 If {ABC} have common factors, then dividing A, B and C by one of the common factors will yield a scaled-down PT. If they are divided by the highest common factor the resulting PT will be a fully scaled-down PT, i.e. a PPT. Similarly, if A, B and C are all multiplied by a common multiplier, it will yield a scaled-up PT which is necessarily non-PPT.

2.6 A Hypotenuse Number (HN) is an integer that can take the value of A in any PT. (Not all integers are HNs. Some of the integers that are HNs are: 5, 10, 13, 15, 17, 20, 25, 26, 29, ... etc).

2.7 If a HN is also a prime number the the HN is termed as a Prime Hypotenuse Number (PHN). (Examples are: 5, 13, 17, 29, ...etc).

3. TRIPLET GENERATORS:

3.1 A pair of distinct integers generate a unique PT.

Let P and Q be two integers of which P > Q
Let U = P^2 + Q^2
Let V = P^2 - Q^2
Let W = 2*P*Q
Then P and Q generate a unique PT {ABC} given by setting
A = U
B = larger of V and W
C = smaller of V and W

3.2 If {ABC} is a PT then its generators P and Q can be found as under:

Let W = A + B

3.2.1 If W is a perfect square then
P = Sqrt((A + C)/2) and Q = Sqrt((A - C)/2)
3.2.2 If W is not a perfect square then
P = Sqrt((A + B)/2) and Q = Sqrt((A - B)/2)

3.3 If P-Q is equal to one, then A-B is also equal to one.

3.4 In general, if P-Q is equal to j, then the difference between the hypotenuse (A) and one of the adjacent sides (either B or C) is equal to j^2. In other words, either A-B or A-C is equal to j^2.

3.4.1 The value of j/P determines the above.
If j/P is less than (2 - sqrt*2) then (A - B) = j^2
If j/P is greater than (2 - sqrt*2) then (A - C) = j^2

3.5 If P and Q are relatively prime to each other and of opposite odd-even parity then they generate a PPT.

4. SOME TRIPLET ELEMENTS RELATIONS:

If A,B and C are the elements of a PT then:

4.1 If B or C is added to A then one results in a perfect square, and the other in twice a perfect square.

4.2 If B or C is subtracted from A then one results in a perfect square, and the other in twice a perfect square.

4.3 At least one element of any PT is divisible by 3, and one by 4, and one by 5.

4.4 The product of the two adjacent sides is always divisible by 12.

4.5 The product of all three sides is always divisible by 60.

5. ADJACENT SIDES:

5.1 Whereas all integers cannot be the hypotenuse of a PT, all integers, except 1 and 2 can be the adjacent side of a PT.

5.2 If G be any integer greater than 2, and designating any one adjacent side of a PT (i.e. either B or C). At least one other integer can be found that will form the other adjacent side of a PT. This can be done as under:

5.2.1 If G is odd:
Resolve G into any two factors - say v and w, where v > w. If

G is a prime number then $v = G$ and $w = 1$.
Let $P = (v + w)/2$ and $Q = (v-w)/2$.
Then P and Q are the generators of the required PT. (See Sec 3.)
If v and w are chosen prime to each other, then it yields a PPT.

5.2.2 If G is even:
Let $G = 2PQ$, where P and Q may be any 2 integers, $P > Q$.
Then P and Q are the generators of the required PT. (See Sec 3.)
If G is even but not divisible by 4, then no primitive solution can exist.
If G is divisible by 4 then choose P and Q relatively prime to each other, and of unlike odd-even parity, to obtain a PPT.

5.3 Since G can be resolved as above in may be more than one ways, G may be an adjacent side in more than one PT.

5.4 To find the number of PTs in which G is one adjacent side:
Resolve G into its prime factors, (and their powers). In general the resolution can be expressed as under:

$$G = p_1^{m_1} . p_2^{m_2} . p_3^{m_3} .... p_r^{m_r}$$

where $p_1, p_2, p_3 .... p_r$ are all prime numbers, and $m_1, m_2, m_3 .... m_r$ are their respective powers.

5.5 The number of primitive triplets in which G can be an adjacent side is given by: $2^{(r-1)}$. Except when G is an even number not divisible by 4, in which case no primitive solutions are possible

5.6 The total number of triplets (prime or otherwise) in which G can be an adjacent side is given by:

5.6.1 If G is odd:
All the prime factors (p_1 to p_r) are odd, and the number of triplets is given by: $\{(2.a_1+1)(2.a_2+1)(2.a_3+1)....(2a_r+1) - 1\}/2$

5.6.2 If G is even:
All the prime factors except one, are odd. The only even prime factor is 2. Let $p_1=2$ and a_1 be the power of p_1. All other prime factors (p_2 to p_r) are odd, and the number of triplets is given by: $\{(2.a_1-1)(2.a_2+1)(2.a_3+1)....(2a_r+1) - 1\}/2$ (note the -ve sign in the first term of this as against +ve sign in the expression above for odd G. All other signs are +ve in both cases)

5.7 Querry: In how many of the above triplets is G the bigger adjacent side?

6. SOME PROPERTIES OF THE HYPOTENUSE

6.1 A HN can always be expressed as the sum of two squares (or a multiple of the sum of two squares. Conversely, any number that can be expressed as a sum of two squares (or a multiple thereof) is a HN. The two numbers in question are the generators of a PT. (see section 3.)

6.2 A HN may be expressed as above in more than one ways. In such cases the HN may yield as many PT as the number of ways it can be so expressed.

6.3 A PHN can always be expressed in the form $4a + 1$. Conversely, all prime numbers of the form $4a + 1$ are HNs.

6.4 All HN have at least one prime factor of the form $4a + 1$.

6.5 A PHN yields only one PT, and that is always a PPT.

6.6 A non-PHN may yield more than one PTs, and some of them may be PPT.

7. DETERMINING THE NUMBER OF TRIPLETS FOR A GIVEN HYPOTENUSE NUMBER

7.1 A HN may be resolved into its prime factors (and their powers). Not all prime factors may be HNs. If A is a HN then the general form of it's prime factorisation is:

A = [p1^a1.p2^a2.p3^a3....pm^am][q1^b1.q2^b2.q3^b3....qn^an]

where all p and q are the prime factors of A, of which the p(s) are HNs and the q(s) are non-HNs. In other words, all p(s) are of the form 2a - 1.

7.2 The number of PTs having any HN A as the hypotenuse is given by the expression:

{(2a1+1)(2a2+1)(2a3+1).........(2am+1) - 1}/2

7.3 If all the prime factors of A are HNs of the form 4a + 1, (i.e. there are no prime factors of the q type), then some of the PTs of section 7.2 are primitive. The total number of PPTs is given by the expression:

2^(m - 1)

8. ASPECT RATIO RELATIONS

8.1 If (ABC} is a PPT whose AR is T, then there exist another PPT with an AR less than T (however small T may be). One such PPT can be found as under:

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  Let X = C + 1 if C is even
  Or  X = C + 2 if C is odd
  Then A' = (X^2 + 1)/2
       B' = (X^2 - 1)/2
       C' = X
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{A'B'C'} is the required PT. If it is not a PPT, scale it down to its primitive form.

8.2 If {ABC} is a PPT whose AR is T, then there exist another PPT with an AR greater than T (however large T may be). One such PPT can be found as under:

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  Let A' = 3A + 2B + 2C
      B' = 2A + 2B +  C
      C' = 2A +  B + 2C
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{A'B'C'} is the required PT. If it is not a PPT, scale it down to its primitive form.

8.3 If {A1B1C1} and {A2B2C2} are two PPTs whose ARs are T1 and T2 respectively then there exist another PPT whose AR lies between T1 and T2 (however close T1 and T2 may be to each other). One such PPT can be found as under:

If (A1 + B1) is not a perfect square then scale up {A1B1C1} by a factor of 2. (i.e. multiply A1,B1 and C1, by 2 and let these be now considered A1,B1 and C1). Do likewise with {A2B2C2}
Let W = Sqrt(A1 + B1)

X = Sqrt(A1 - B1)
Y = Sqrt(A2 + B2)
Z = Sqrt(A2 - B2)
Then A' = A1 + A2 + WY + XZ
B' = B1 + B2 + WY - XZ
C' = C1 + C2 + XY + WZ

{A'B'C'} is the required PT. If it is not a PPT, scale it down to its primitive form.

9. TABLE OF ALL TRIPLETS WITH HYPOTENUSE UPTO 1000:
(PPT are in brackets)

(5 4 3)	10 8 6	(13 12 5)	15 12 9	(17 15 8)
20 16 12	25 20 15	(25 24 7)	26 24 10	(29 21 20)
30 24 18	34 30 16	35 28 21	(37 35 12)	39 36 15
40 32 24	(41 40 9)	45 36 27	50 40 30	50 48 14
51 45 24	52 48 20	(53 45 28)	55 44 33	58 42 40
60 48 36	(61 60 11)	65 52 39	(65 56 33)	65 60 25
(65 63 16)	68 60 32	70 56 42	(73 55 48)	74 70 24
75 60 45	75 72 21	78 72 30	80 64 48	82 80 18
85 68 51	85 75 40	(85 77 36)	(85 84 13)	87 63 60
(89 80 39)	90 72 54	91 84 35	95 76 57	(97 72 65)
100 80 60	100 96 28	(101 99 20)	102 90 48	104 96 40
105 84 63	106 90 56	(109 91 60)	110 88 66	111 105 36
(113 112 15)	115 92 69	116 84 80	117 108 45	119 105 56
120 96 72	122 120 22	123 120 27	125 100 75	(125 117 44)
125 120 35	130 104 78	130 112 66	130 120 50	130 126 32
135 108 81	136 120 64	(137 105 88)	140 112 84	143 132 55
145 105 100	145 116 87	(145 143 24)	(145 144 17)	146 110 96
148 140 48	(149 140 51)	150 120 90	150 144 42	153 135 72
155 124 93	156 144 60	(157 132 85)	159 135 84	160 128 96
164 160 36	165 132 99	(169 120 119)	169 156 65	170 136 102
170 150 80	170 154 72	170 168 26	(173 165 52)	174 126 120
175 140 105	175 168 49	178 160 78	180 144 108	(181 180 19)
182 168 70	183 180 33	185 148 111	(185 153 104)	185 175 60
(185 176 57)	187 165 88	190 152 114	(193 168 95)	194 144 130
195 156 117	195 168 99	195 180 75	195 189 48	(197 195 28)
200 160 120	200 192 56	202 198 40	203 147 140	204 180 96
(205 156 133)	205 164 123	(205 187 84)	205 200 45	208 192 80
210 168 126	212 180 112	215 172 129	218 182 120	219 165 144
220 176 132	(221 171 140)	221 195 104	221 204 85	(221 220 21)
222 210 72	225 180 135	225 216 63	226 224 30	(229 221 60)
230 184 138	232 168 160	(233 208 105)	234 216 90	235 188 141
238 210 112	240 192 144	(241 209 120)	244 240 44	245 196 147
246 240 54	247 228 95	250 200 150	250 234 88	250 240 70
255 204 153	255 225 120	255 231 108	255 252 39	(257 255 32)
259 245 84	260 208 156	260 224 132	260 240 100	260 252 64
261 189 180	265 212 159	265 225 140	(265 247 96)	(265 264 23)
267 240 117	(269 260 69)	270 216 162	272 240 128	273 252 105
274 210 176	275 220 165	275 264 77	(277 252 115)	280 224 168
(281 231 160)	285 228 171	286 264 110	287 280 63	(289 240 161)
289 255 136	290 210 200	290 232 174	290 286 48	290 288 34
291 216 195	292 220 192	(293 285 68)	295 236 177	296 280 96
298 280 102	299 276 115	300 240 180	300 288 84	303 297 60
(305 224 207)	305 244 183	(305 273 136)	305 300 55	306 270 144
310 248 186	312 288 120	(313 312 25)	314 264 170	315 252 189
(317 308 75)	318 270 168	319 231 220	320 256 192	323 285 152
(325 253 204)	325 260 195	325 280 165	325 300 125	325 312 91
325 315 80	(325 323 36)	327 273 180	328 320 72	330 264 198
333 315 108	335 268 201	(337 288 175)	338 240 238	338 312 130
339 336 45	340 272 204	340 300 160	340 308 144	340 336 52
345 276 207	346 330 104	348 252 240	(349 299 180)	350 280 210
350 336 98	351 324 135	(353 272 225)	355 284 213	356 320 156

357 315 168	360 288 216	362 360 38	364 336 140	365 275 240
365 292 219	(365 357 76)	(365 364 27)	366 360 66	369 360 81
370 296 222	370 306 208	370 350 120	370 352 114	371 315 196
(373 275 252)	374 330 176	375 300 225	375 351 132	375 360 105
377 273 260	(377 345 152)	377 348 145	(377 352 135)	380 304 228
385 308 231	386 336 190	388 288 260	(389 340 189)	390 312 234
390 336 198	390 360 150	390 378 96	391 345 184	394 390 56
395 316 237	(397 325 228)	400 320 240	400 384 112	(401 399 40)
403 372 155	404 396 80	405 324 243	406 294 280	407 385 132
408 360 192	(409 391 120)	410 312 266	410 328 246	410 374 168
410 400 90	411 315 264	415 332 249	416 384 160	420 336 252
(421 420 29)	424 360 224	(425 304 297)	425 340 255	425 375 200
425 385 180	425 408 119	(425 416 87)	425 420 65	427 420 77
429 396 165	430 344 258	(433 408 145)	435 315 300	435 348 261
435 429 72	435 432 51	436 364 240	438 330 288	440 352 264
442 342 280	442 390 208	442 408 170	442 440 42	444 420 144
445 356 267	(445 396 203)	445 400 195	(445 437 84)	447 420 153
(449 351 280)	450 360 270	450 432 126	451 440 99	452 448 60
455 364 273	455 392 231	455 420 175	455 441 112	(457 425 168)
458 442 120	459 405 216	460 368 276	(461 380 261)	464 336 320
465 372 279	466 416 210	468 432 180	470 376 282	471 396 255
475 380 285	475 456 133	476 420 224	477 405 252	480 384 288
(481 360 319)	481 444 185	481 455 156	(481 480 31)	482 418 240
485 360 325	485 388 291	(485 476 93)	(485 483 44)	488 480 88
490 392 294	492 480 108	493 357 340	493 435 232	(493 468 155)
(493 475 132)	494 456 190	495 396 297	500 400 300	500 468 176
500 480 140	(505 377 336)	505 404 303	(505 456 217)	505 495 100
507 360 357	507 468 195	(509 459 220)	510 408 306	510 450 240
510 462 216	510 504 78	511 385 336	514 510 64	515 412 309
518 490 168	519 495 156	520 416 312	520 448 264	520 480 200
520 504 128	(521 440 279)	522 378 360	525 420 315	525 504 147
527 465 248	530 424 318	530 450 280	530 494 192	530 528 46
(533 435 308)	533 492 205	533 520 117	(533 525 92)	534 480 234
535 428 321	538 520 138	540 432 324	(541 420 341)	543 540 57
544 480 256	545 436 327	545 455 300	(545 513 184)	(545 544 33)
546 504 210	548 420 352	549 540 99	550 440 330	550 528 154
551 399 380	554 504 230	555 444 333	555 459 312	555 525 180
555 528 171	(557 532 165)	559 516 215	560 448 336	561 495 264
562 462 320	(565 403 396)	565 452 339	(565 493 276)	565 560 75
(569 520 231)	570 456 342	572 528 220	574 560 126	575 460 345
575 552 161	(577 575 48)	578 480 322	578 510 272	579 504 285
580 420 400	580 464 348	580 572 96	580 576 68	582 432 390
583 495 308	584 440 384	585 468 351	585 504 297	585 540 225
585 567 144	586 570 136	590 472 354	591 585 84	592 560 192
(593 465 368)	595 476 357	595 525 280	595 539 252	595 588 91
596 560 204	598 552 230	600 480 360	600 576 168	(601 551 240)
605 484 363	606 594 120	609 441 420	610 448 414	610 488 366
610 546 272	610 600 110	611 564 235	612 540 288	(613 612 35)
615 468 399	615 492 369	615 561 252	615 600 135	(617 608 105)
620 496 372	623 560 273	624 576 240	625 500 375	(625 527 336)
625 585 220	625 600 175	626 624 50	628 528 340	(629 460 429)
629 555 296	629 595 204	(629 621 100)	630 504 378	634 616 150
635 508 381	636 540 336	637 588 245	638 462 440	640 512 384
(641 609 200)	645 516 387	646 570 304	650 506 408	650 520 390
650 560 330	650 600 250	650 624 182	650 630 160	650 646 72
(653 572 315)	654 546 360	655 524 393	656 640 144	657 495 432
660 528 396	(661 589 300)	663 513 420	663 585 312	663 612 255
663 660 63	665 532 399	666 630 216	667 483 460	670 536 402
671 660 121	(673 552 385)	674 576 350	675 540 405	675 648 189
676 480 476	676 624 260	(677 675 52)	678 672 90	679 504 455
680 544 408	680 600 320	680 616 288	680 672 104	685 525 440
685 548 411	(685 667 156)	(685 684 37)	687 663 180	(689 561 400)
689 585 364	689 636 265	(689 680 111)	690 552 414	692 660 208
695 556 417	696 504 480	(697 528 455)	697 615 328	(697 672 185)

697 680 153	698 598 360	699 624 315	700 560 420	700 672 196
(701 651 260)	702 648 270	703 665 228	705 564 423	706 544 450
707 693 140	(709 660 259)	710 568 426	712 640 312	714 630 336
715 572 429	715 616 363	715 660 275	715 693 176	720 576 432
723 627 360	724 720 76	725 525 500	725 580 435	(725 627 364)
725 644 333	725 696 203	725 715 120	725 720 85	728 672 280
730 550 480	730 584 438	730 714 152	730 728 54	731 645 344
732 720 132	(733 725 108)	735 588 441	738 720 162	740 592 444
740 612 416	740 700 240	740 704 228	741 684 285	742 630 392
745 596 447	(745 624 407)	745 700 255	(745 713 216)	746 550 504
748 660 352	750 600 450	750 702 264	750 720 210	754 546 520
754 690 304	754 696 290	754 704 270	755 604 453	(757 595 468)
760 608 456	(761 760 39)	763 637 420	765 612 459	765 675 360
765 693 324	765 756 117	767 708 295	(769 600 481)	770 616 462
771 765 96	772 672 380	(773 748 195)	775 620 465	775 744 217
776 576 520	777 735 252	778 680 378	779 760 171	780 624 468
780 672 396	780 720 300	780 756 192	782 690 368	783 567 540
785 628 471	785 660 425	(785 736 273)	(785 783 56)	788 780 112
790 632 474	791 784 105	(793 665 432)	793 732 305	(793 775 168)
793 780 143	794 650 456	795 636 477	795 675 420	795 741 288
795 792 69	(797 572 555)	799 705 376	800 640 480	800 768 224
801 720 351	802 798 80	803 605 528	805 644 483	806 744 310
807 780 207	808 792 160	(809 759 280)	810 648 486	812 588 560
814 770 264	815 652 489	816 720 384	818 782 240	819 756 315
820 624 532	820 656 492	820 748 336	820 800 180	(821 700 429)
822 630 528	825 660 495	825 792 231	(829 629 540)	830 664 498
831 756 345	832 768 320	833 735 392	835 668 501	840 672 504
841 609 580	(841 840 41)	842 840 58	843 693 480	845 600 595
845 676 507	845 728 429	845 780 325	845 819 208	(845 836 123)
(845 837 116)	848 720 448	850 608 594	850 680 510	850 750 400
850 770 360	850 816 238	850 832 174	850 840 130	851 805 276
(853 828 205)	854 840 154	855 684 513	(857 825 232)	858 792 330
860 688 516	861 840 189	865 692 519	(865 703 504)	(865 816 287)
865 825 260	866 816 290	867 720 483	867 765 408	870 630 600
870 696 522	870 858 144	870 864 102	871 804 335	872 728 480
873 648 585	875 700 525	875 819 308	875 840 245	876 660 576
(877 805 348)	879 855 204	880 704 528	(881 800 369)	884 684 560
884 780 416	884 816 340	884 880 84	885 708 531	888 840 288
890 712 534	890 792 406	890 800 390	890 874 168	894 840 306
895 716 537	897 828 345	898 702 560	899 651 620	900 720 540
900 864 252	901 765 476	(901 780 451)	901 795 424	(901 899 60)
902 880 198	904 896 120	(905 663 616)	905 724 543	(905 777 464)
905 900 95	909 891 180	910 728 546	910 784 462	910 840 350
910 882 224	914 850 336	915 672 621	915 732 549	915 819 408
915 900 165	916 884 240	918 810 432	920 736 552	922 760 522
923 852 355	925 740 555	(925 756 533)	925 765 520	925 875 300
925 880 285	925 888 259	(925 924 43)	928 672 640	(929 920 129)
930 744 558	932 832 420	935 748 561	935 825 440	935 847 396
935 924 143	936 864 360	(937 912 215)	939 936 75	940 752 564
(941 741 580)	942 792 510	943 920 207	945 756 567	949 715 624
(949 851 420)	949 876 365	(949 900 301)	950 760 570	950 912 266
951 924 225	952 840 448	(953 728 615)	954 810 504	955 764 573
957 693 660	959 735 616	960 768 576	962 720 638	962 888 370
962 910 312	962 960 62	964 836 480	965 772 579	965 840 475
(965 884 387)	(965 957 124)	969 855 456	970 720 650	970 776 582
970 952 186	970 966 88	975 759 612	975 780 585	975 840 495
975 900 375	975 936 273	975 945 240	975 969 108	976 960 176
(977 945 248)	979 880 429	980 784 588	981 819 540	984 960 216
(985 697 696)	985 788 591	(985 864 473)	985 975 140	986 714 680
986 870 464	986 936 310	986 950 264	988 912 380	990 792 594
995 796 597	(997 925 372)	999 945 324	1000 800 600	1000 936 352
1000 960 280				